# Periodically Correlated Processes

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# Introduction

**DEFINITION 1.1** *The stochastic process* $\{X(t,\omega), t \in I\}$ *defined on the probability space* $L_2(\Omega, F, P)$ *is called periodically correlated (PC) with period* $T$ *if* $T$ *is the smallest integer such that:*

$$E\{X(u)\} = m(u) = m(u+T),$$

$$E\{(X(u)-m(u))\overline{(X(v)-m(v))}\} = R(u,v) = R(u+T, v+T),$$

*for every* $(u,v) \in I \times I$.

A stationary sequence is periodically correlated with $T = 1$.

# Introduction cont.

- PC processes are a class of processes which are in general nonstationary but exhibit many of the properties of stationary processes
- PC processes have been used as models in meteorology, radio physics and communications engineering
- PC processes have been called periodically nonstationary, cyclostationary, periodically stationary and processes with periodic structure

# Simple models for PC sequences

- If $X(t) \in L_2(\Omega, F, P)$ is a periodic sequence with period $T$, then $X(t)$ is PC.
- If $X(t) \in L_2(\Omega, F, P)$ is wide sense stationary with $E\{X(t)\} = 0$ and $f(t)$ is a scalar periodic sequence $f(t) = f(t+T)$, then

$$Y(t) = f(t) + X(t)$$

  is PC with period T.

# Simple models for PC sequences cont.

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$$Y(t) = f(t)X(t)$$

  is PC with period T.

- If $X(t) \in L_2(\Omega, F, P)$ is wide sense stationary with $E\{X(t)\} = 0$ and $f(t)$ is a scalar periodic sequence $f(t) = f(t+T)$ taking values in the index set, then

$$Y(t) = X(t + f(t))$$

  is PC with period T.

# Spectral representation of PC processes

Gladyshev (1961) showed that the covariance $R_X(m,n)$ of PC processes has representation

$$R_X(m,n) = \int_0^{2\pi}\int_0^{2\pi} e^{im\omega_1 - in\omega_2} f_Z(\omega_1,\omega_2)d\omega_1 d\omega_2 \qquad (1)$$

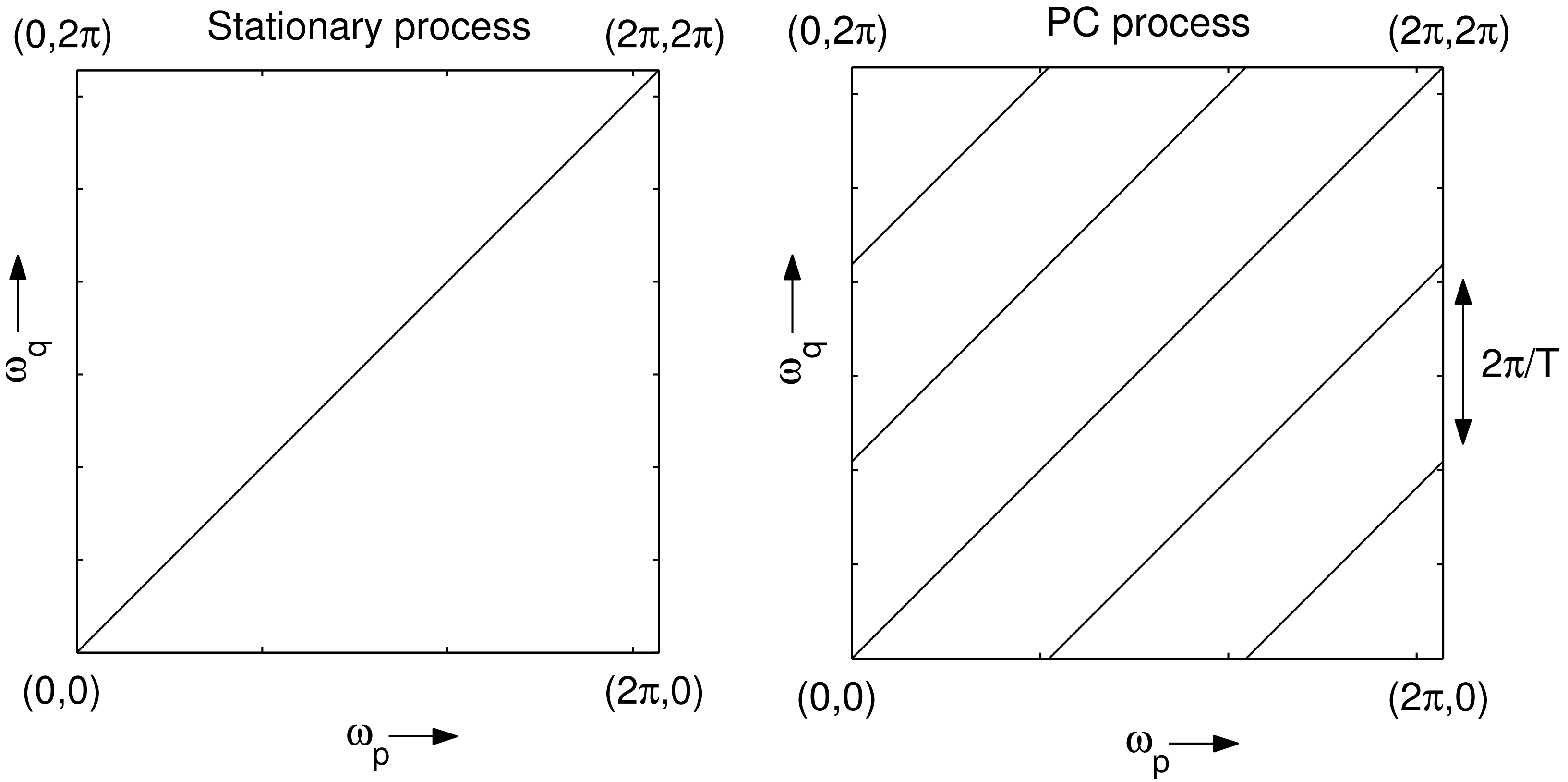
The support of spectral density $f_Z(\omega_1,\omega_2)$ is then contained in the set of $2[T]-1$ diagonal lines

$$S = \bigcup_{k\in\mathbb{Z}}\{[\omega_1,\omega_2] \in [0,2\pi)\times[0,2\pi) : \omega_1 = \omega_2 + 2k\pi/T\}$$

where T is the period of the correlation function. It follows that the spectral density function is given by

$$f_Z(\omega_1,\omega_2) = \begin{cases} f_Z(\omega_1,\omega_2) & [\omega_1,\omega_2] \in S \\ 0 & [\omega_1,\omega_2] \notin S \end{cases}$$

# Spectral representation of PC processes cont.



# Measuring periodic correlation

To investigate the presence of periodic correlation we can use a statistic which is called ***sample coherence*** and is defined as

$$|\hat{\gamma}(p,q,M)|^2 = \frac{|\sum_{m=0}^{M-1} I_N(\omega_{p+m})\overline{I_N(\omega_{q+m})}|^2}{\sum_{m=0}^{M-1}|I_N(\omega_{p+m})|^2 \sum_{m=0}^{M-1}|I_N(\omega_{q+m})|^2}, \qquad (2)$$

- $0 < p, q \le N$, $N$ is the sample length
- $M$ is the smoothness coefficient

It takes only real values between 0 and 1. For PC processes the values taken on the support lines are significantly different from those taken for the intermediate frequencies.

# Measuring periodic correlation cont.

There are two standard graphical tools for detecting periodic correlation. Both are based on sample coherence.

- The ***coherent statistic*** is defined as $|\hat{\gamma}(0, d, N)|^2$, where $d = |q - p|$.
- The ***incoherent statistic*** is given by

$$\delta(d, M) = \frac{1}{L+1} \sum_{p=0}^{L} |\hat{\gamma}(pM, pM + d, M)|^2 \ ,$$

where $L = \left[\frac{N-1-d}{M}\right]$.

Peaks in one-dimensional plots of both statistics indicate periods of length $T = 1/d$.

# Measuring periodic correlation cont.

We propose the ***measure of fitness (MoF)*** statistic

$$MoF(d, M) = \frac{1}{N} \sum_{p=1}^{N} \kappa_\alpha(p, p + d, M)$$

where

- $\kappa_\alpha(p, q, M) = \begin{cases} 1 & |\hat{\gamma}(p, q, M)|^2 \geq \hat{c}_\alpha \\ 0 & |\hat{\gamma}(p, q, M)|^2 < \hat{c}_\alpha \end{cases}$
- $\alpha$ is the confidence level
- $\hat{c}_\alpha$ is the estimator of the critical value

The estimator $\hat{c}_\alpha$ is computed using the ***Moving Blocks Bootstrap(MBB)*** procedure

# Measuring periodic correlation cont.

For a sample $\{X_1, \ldots, X_N\}$ the method for computing estimator $\hat{c}_\alpha$ based on the MBB procedure consists of the following

- denote blocks of length $b$ as $B_i = (X_i, \ldots, X_{i+b-1})$ , $i = 1, \ldots, N-b+1$
- draw from the set $\{B_i : i = 1, \ldots, N-b+1\}$ a number $k = [N/b]$ of blocks and construct the bootstrap sample $\{X_1^*, \ldots, X_l^*\} = \{B_1^*, \ldots, B_k^*\}$ by gluing the blocks together
- compute the coherence statistic $|\hat{\gamma}_i^*(p, q, M)|^2$ at each point $(p, q)$, B times for each bootstrap sample
- approximate the critical value $\hat{c}_\alpha$ as the $1-\alpha$ quantile of the bootstrap distribution. It is the $[\alpha \cdot N]$-th largest value of the sequence $\{|\hat{\gamma}_1^*(p, q, M)|^2, \ldots, |\hat{\gamma}_B^*(p, q, M)|^2\}$

# Example 1

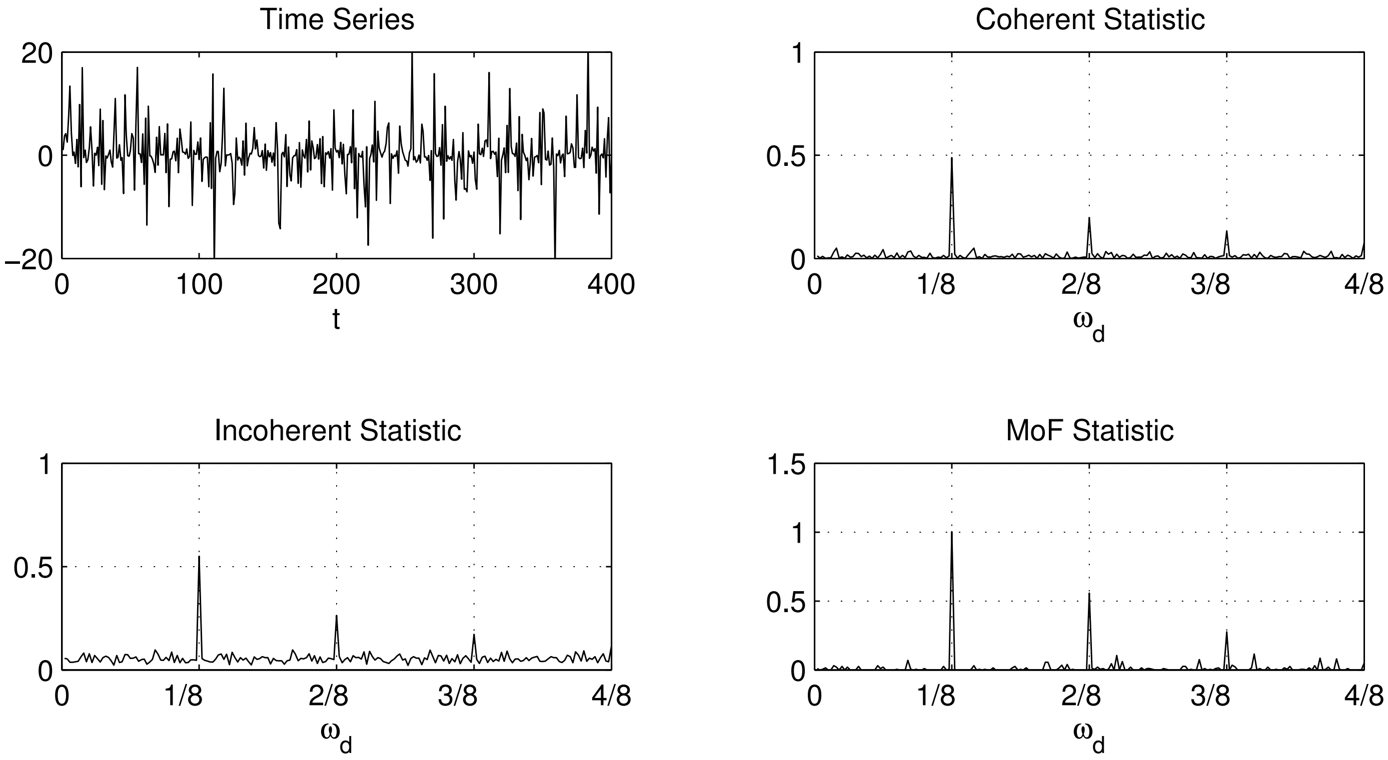



Figure 1: PC process $X_n = S_n \cdot f^2(n)$, where $f(n) = \mathrm{mod}(n, 8)$. The test parameters: $M$=20, $B$=100, $\alpha$=0.01, $N$=400.

# Example 2

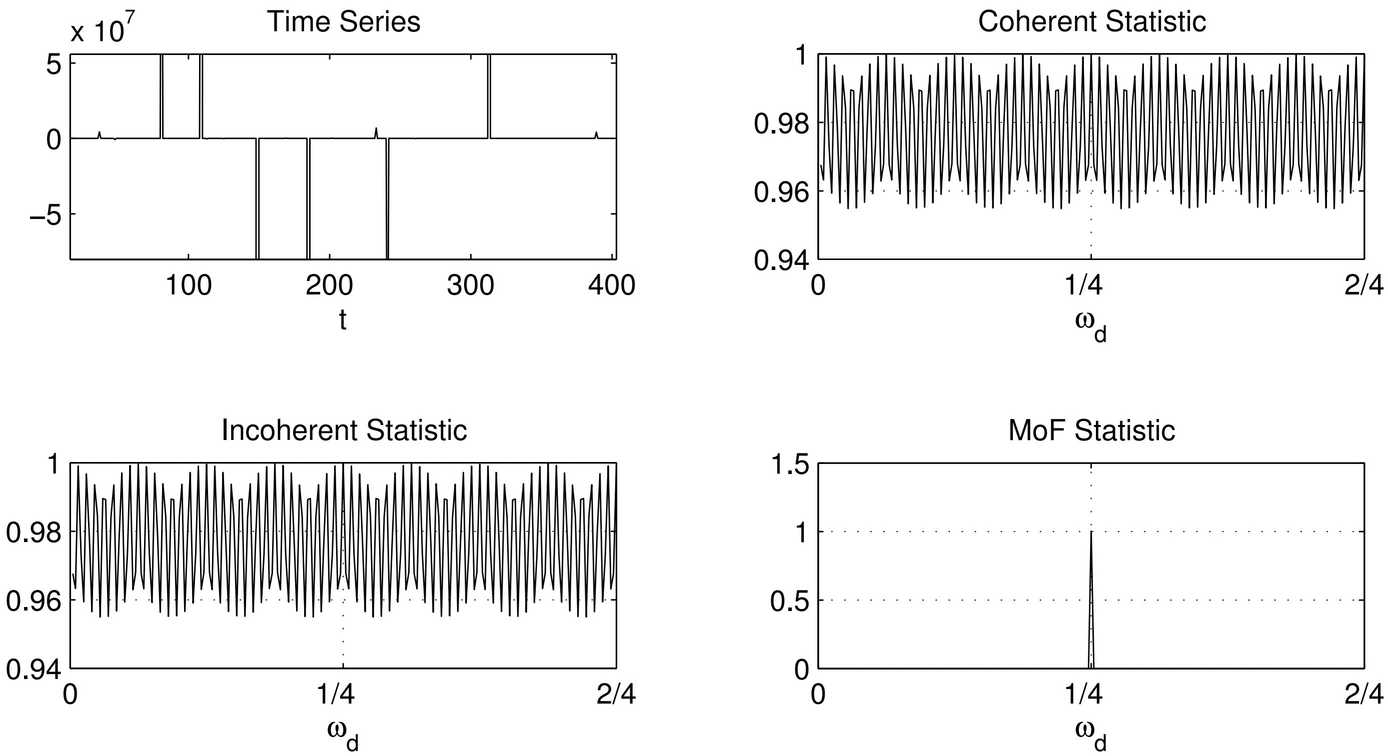



Figure 2: PC process $X_n = S_n \cdot \exp[8 \cdot \{1 + \sin(\frac{\pi n}{2}) \cdot \frac{4}{5}\xi_n\}]$. The test parameters: $M$=20, $B$=100, $\alpha$=0.01, $N$=400.

# Example 3

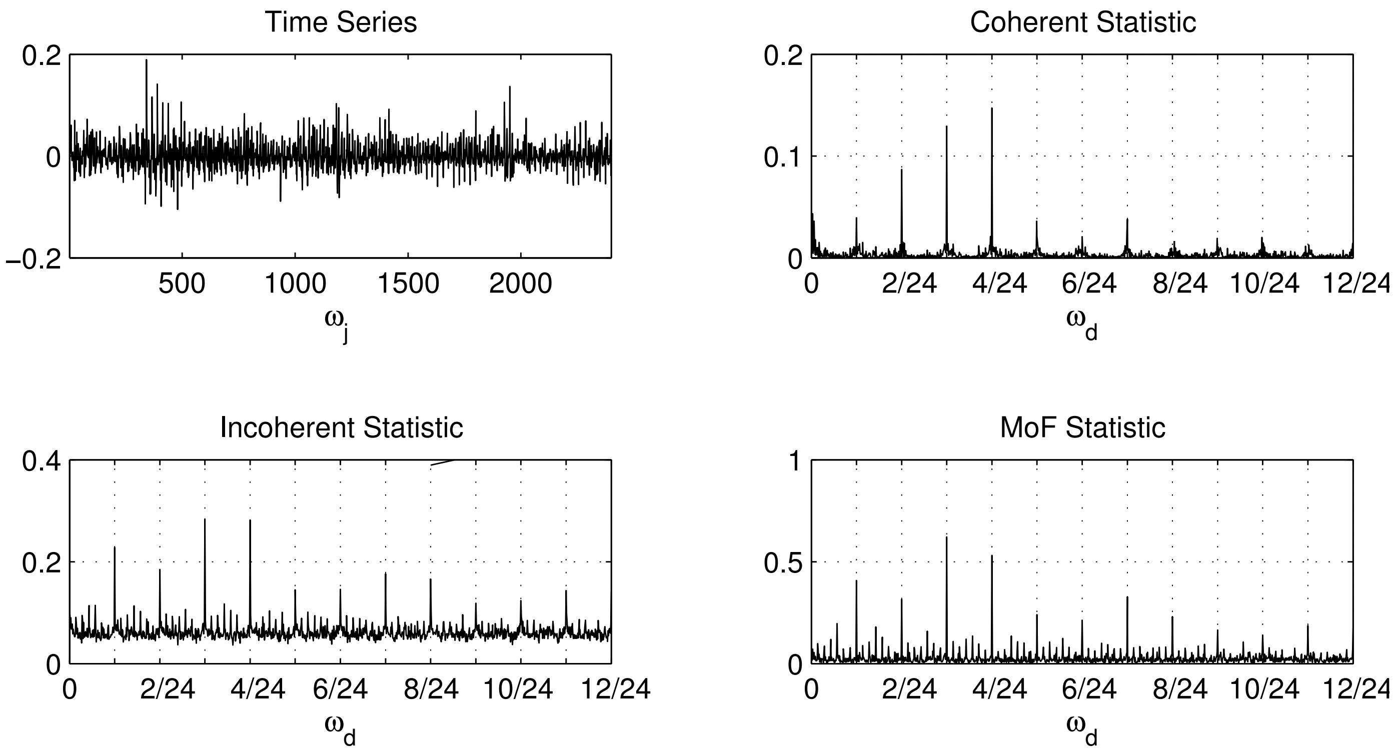



Figure 3: Hourly returns of Nord Pool electricity spot prices during the period January 1st, 1997 – December 25th, 2001(100 days). The test parameters: $M$=20, $B$=100 and $\alpha$=0.01, $N$=2400.

# Example 4

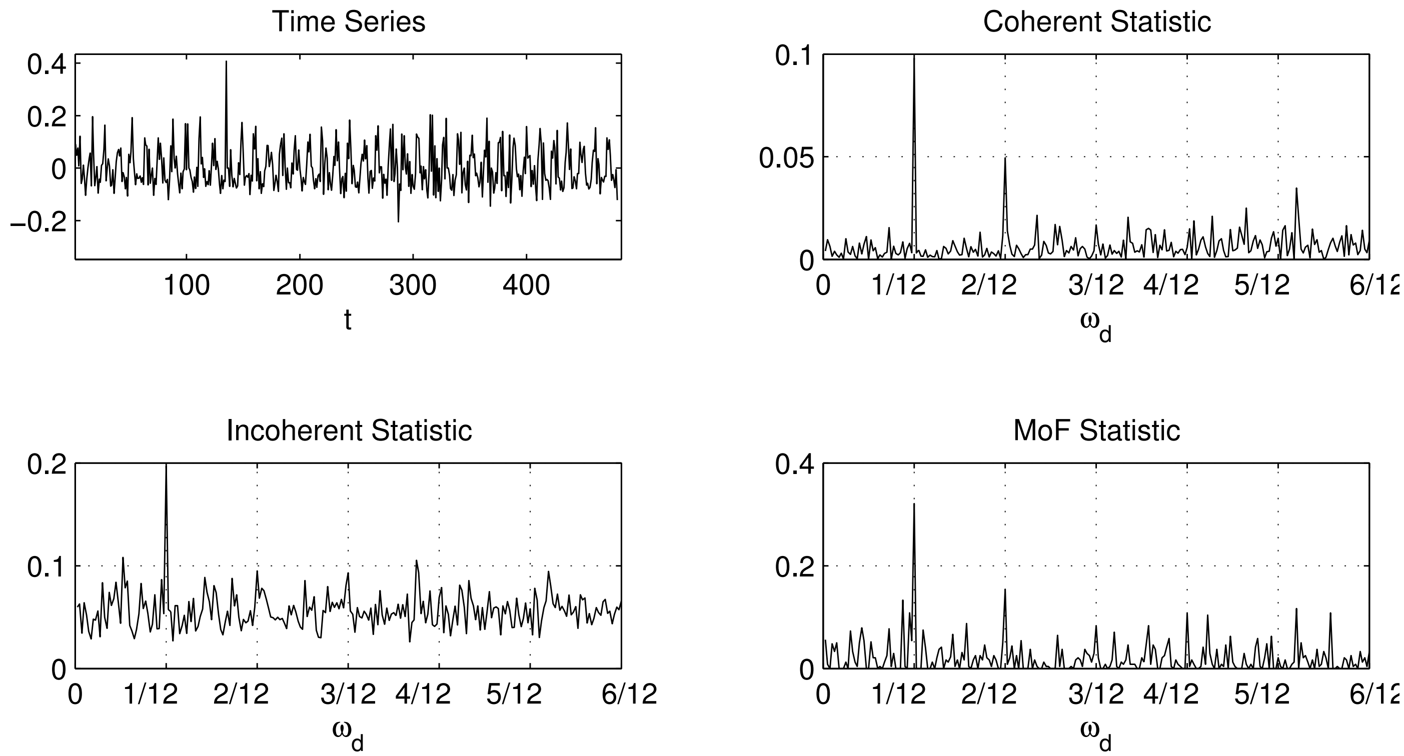



Figure 4: Monthly AROSA Switzerland Ozone levels. The test parameters: $M$=20, $B$=100 and $\alpha$=0.01, $N$=480.

# Example 5

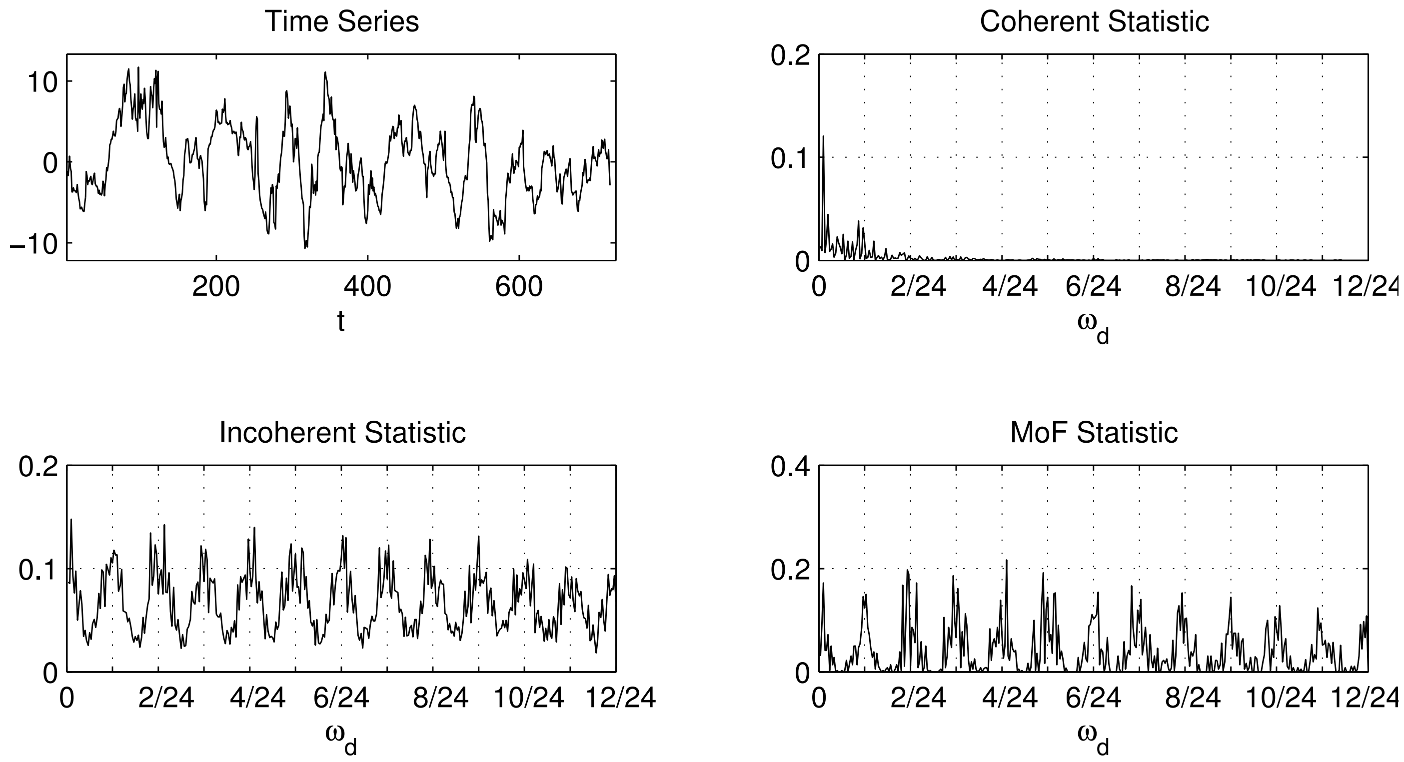



Figure 5: Hourly temperatures in Wrocław from January 1993. The test parameters: $M{=}20$, $B{=}100$ and $\alpha{=}0.01$, $N{=}720$.

# Example 6

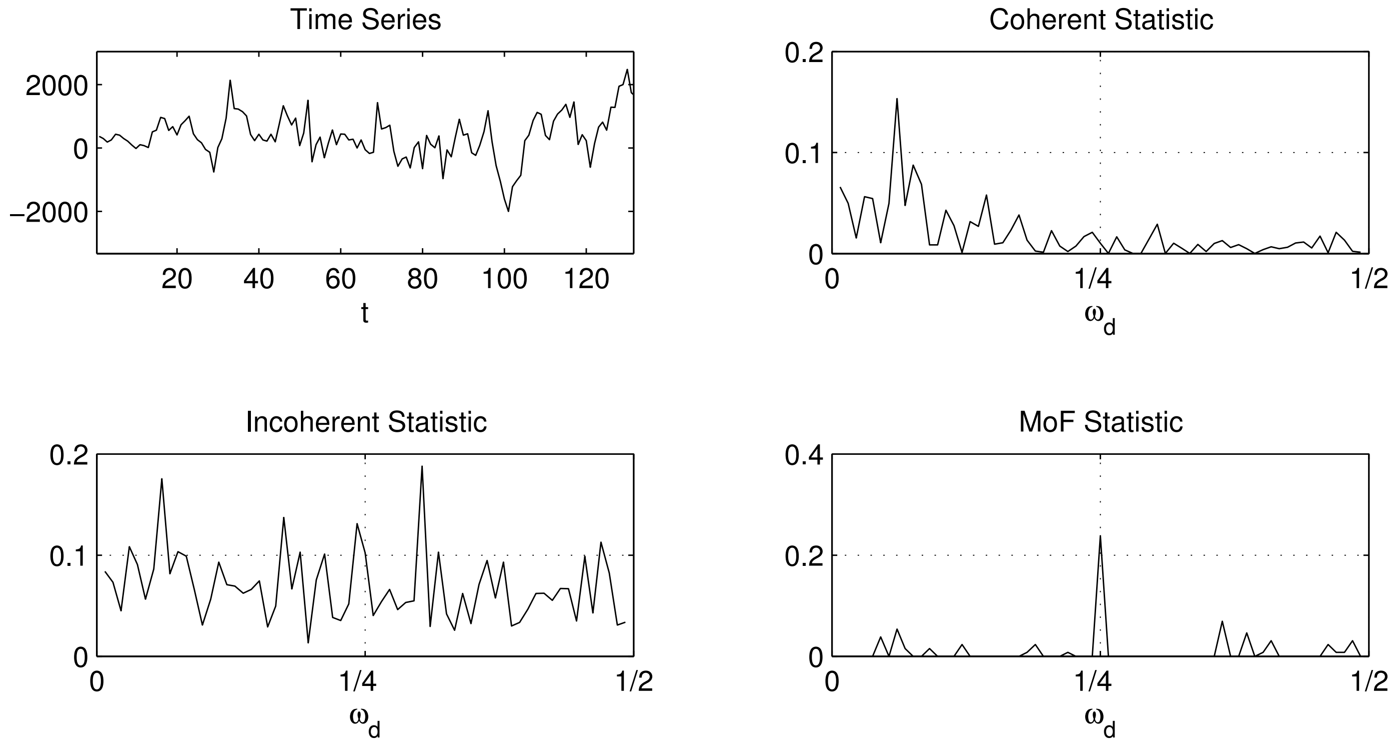



Figure 6: Quarterly real total investment in the United Kingdom 1955-1988. The test parameters: $M$=20, $B$=100 and $\alpha$=0.01, $N$=138.

# PARMA models cont.

**DEFINITION 1.2** *The system PARMA(p,q) is defined as:*

$$X(n) - \sum_{k=1}^{p} \phi_k(n) X(n-k) = \sum_{k=0}^{q-1} \theta_k(n) \xi_{n-k}, \tag{3}$$

*where the sequences* $\{\phi_k(n)\}$ *and* $\{\theta_k(n)\}$ *are periodic in* $n$ *with the same period* $T$ *and* $\xi_k$ *is uncorrelated sequence of random variables with mean* $0$ *and unit variance.*

# PARMA vs. VARMA

We arrange coefficients of the left and right-hand sides of (3) into $T \times (l+1)T$ i $T \times (r+1)T$ matrices as follows

$$\begin{bmatrix} 1 & -\phi_1(0) & -\phi_2(0) & \dots & -\phi_p(0) & 0 & \dots & 0 \\ 0 & 1 & -\phi_1(1) & \dots & -\phi_{p-1}(1) & -\phi_p(1) & \dots & 0 \\ 0 & 0 & 1 & -\phi_1(2) & \dots & -\phi_{p-2}(2) & \dots & 0 \\ & \vdots & & & \vdots & & & \\ 0 & 0 & \dots & 1 & -\phi_1(T-1) & \dots & -\phi_p(T-1) & \dots \end{bmatrix}$$

# PARMA vs. VARMA cont.

$$\begin{bmatrix}
\theta_0(0) & \theta_1(0) & \dots & \theta_q(0) & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & \theta_0(1) & \theta_1(1) & \dots & \theta_q(1) & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & \theta_0(2) & \theta_1(2) & \dots & \theta_q(2) & 0 & 0 & \dots & 0 \\
 & \vdots & & & & & \vdots & & & \\
0 & 0 & 0 & \dots & 0 & \theta_1(T-1) & \dots & \theta_q(T-1) & \dots &
\end{bmatrix}.$$

# PARMA vs. VARMA cont.

We denote the consecutive $T \times T$ blocks of the two matrices by $\Phi_0, \ldots, \Phi_l$ and $\Theta_0, \ldots, \Theta_r$, respectively, then (3) can be written as the T-dimensional vector ARMA (VARMA) system

$$\mathbf{X}(n) - \sum_{k=1}^{l} \Phi_k \mathbf{X}(n-k) = \sum_{k=0}^{r} \Theta_k \Xi_{n-k} \qquad (4)$$

where

$$\mathbf{X}(n) = [x(nT), x(nT-1), \ldots, x((n-1)T+1)]^t$$

and

$$\Xi_n = [\xi_{nT}, \xi_{nT-1}, \ldots, \xi_{(n-1)T+1}]^t.$$

# Example 7

We consider PARMA(1,1) model with the period $T = 2$ given by the formula:

$$X(n) - \phi(n)X(n-1) = \theta_0(n)\xi_n + \theta_1(n)\xi_{n-1}.$$

# Example 7 cont.

Then the adequate VARMA model has the following matrices of coefficients:

$$\Phi_1 = \begin{bmatrix} 1 & -\phi(0) \\ 0 & 1 \end{bmatrix}$$

$$\Phi_2 = \begin{bmatrix} 0 & 0 \\ -\phi(1) & 0 \end{bmatrix}$$

$$\Theta_0 = \begin{bmatrix} \theta_0(0) & \theta_1(0) \\ 0 & \theta_0(1) \end{bmatrix}$$

$$\Theta_1 = \begin{bmatrix} 0 & 0 \\ \theta_1(1) & 0 \end{bmatrix}.$$

# ARMA(p,q) models with varying coefficients

**DEFINITION 1.3** *An ARMA(p,q) system with varying coefficients is a system of linear equations*

$$X(n) - \sum_{k=1}^{p} b_k(n) X(n-k) = \sum_{k=0}^{q-1} a_k(n) \xi_{n-k}, \tag{5}$$

*where where* $(b_k(n))$, $k = 1, \ldots, p$ *and* $(a_k(n))$, $k = 0, \ldots, q-1$, *are sequences of complex numbers, and* $(\xi_n)$ *is a sequence of uncorrelated complex random variables with mean zero and unit variances.*

# ARMA(p,q) models with varying coefficients cont.

Let

$$B_n = \begin{bmatrix} b_1(n) & b_2(n) & \dots & b_{p-1}(n) & b_p(n) \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

$$\mathbf{B}_k^n = B_n B_{n-1} ... B_k, \qquad n \geq k$$

# ARMA(p,q) models with varying coefficients cont.

$\mathbf{X}(n) = [X(n), X(n-1), \ldots, X(n-p+1)]$ ‘

$Y_n = \left[\sum_{p=0}^{q-1} a_p(n)\xi_{n-p}, 0, \ldots, 0\right]$ ‘

$\Xi_n = [\xi_n, 0, \ldots, 0]'$

$$\mathbf{a}_j(n) = \begin{bmatrix} a_j(n) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

# ARMA(p,q) models with varying coefficients cont.

Then system (5) has the equivalent form:

$$\mathbf{X}(n) - B_n\mathbf{X}(n-1) = \mathbf{Y}_n. \tag{6}$$

The following theorem gives the sufficient and necessary conditions for existence of a unique bounded solution of equation (6).

# Unique and bounded solutions of ARMA(p,q) models with varying coefficients

**THEOREM 1.1** *System (6) has a unique bounded solution iff:*

**I.** (i) $\sup_{n\geq 1} ||\mathbf{B}_1^n|| = \infty$ *and*

(ii) $\sup_{n\in Z} \sum_{s=2-q}^{\infty} \left\| \sum_{j=\max(1,s)}^{q-1+s} (\mathbf{B}_{n+1}^{n+j})^{-1} \mathbf{a}_{j-s}(n+j) \right\|^2 < \infty,$

*or*

**II.** (i) $\sup_{n\leq -1} ||\mathbf{B}_n^0||^{-1} = \infty$ *and*

(ii) $\sup_{n\in Z} \sum_{s=0}^{\infty} \left\| \sum_{j=-s}^{\min(0,q-1-s)} \mathbf{B}_{n+j+1}^{n} \mathbf{a}_{s+j}(n+j) \right\|^2 < \infty.$

# Unique and bounded solutions of ARMA(p,q) models with varying coefficients cont.

**THEOREM 1.2** *If I is satisfied then the solution has the form*

$$\mathbf{X}(n) = -\sum_{s=2-q}^{\infty}\left[\sum_{j=\max(1,s)}^{q-1+s}(\mathbf{B}_{n+1}^{n+j})^{-1}\mathbf{a}_{j-s}(n+j)\right]\Xi_{n+s}.$$

*If II is satisfied then the solution is given by*

$$\mathbf{X}(n) = \sum_{s=0}^{\infty}\left[\sum_{j=-s}^{\min(0,q-1-s)}\mathbf{B}_{n+j+1}^{n}\mathbf{a}_{s+j}(n+j)\right]\Xi_{n-s}.$$

# Unique and bounded solutions of ARMA(p,q) models with varying coefficients cont.

**THEOREM 1.3** *Let* $P = B_1 B_2 \ldots B_T$. *System* (5) *has a bounded solution iff* $||P|| \neq 1$. *The solution is unique and periodically correlated with period* $T$.

# Yule-Walker equation for ARMA(p,q) models

**THEOREM 1.4** *(Yule-Walker equation)*
*Let* $\{X(n), n \in Z\}$ *be the solution of system ARMA(p,q) with varying coefficients and* $||P|| = ||\mathbf{B}_1^T|| < 1$, *then the correlation function satisfies the Yule-Walker equation:*

$$R(n,m) - \sum_{j=1}^{p} b_j(n) R(n-j,m) = \sum_{j=0}^{q-1} a_j(n) U_{m-n+j}(m),$$

*where* $U_j(n)$ *is the first coefficients of the following vector*

$$L_s(n) = \sum_{j=-s}^{\min(0,q-1-s)} \mathbf{B}_{n+j+1}^{n} \mathbf{a}_{s+j}(n+j)[1 \quad 0 \quad 0 \ldots 0]'.$$

# Example 8

We consider the Nord Pool data of price of energy. We fit the best PAR(p) model

$$X(n) - \sum_{j=1}^{p} b_j(n) X(n-j) = \xi_n.$$

The best PAR(p) model with respect to Bayesian information criterion is the PAR(5) model. The $P$ matrix has the following form

$$P = \begin{bmatrix} 3.3689 & -2.5041 & 0.7109 & 0.0821 & -0.0315 \\ 3.3866 & -2.5174 & 0.7147 & 0.0825 & -0.0317 \\ 3.4027 & -2.5271 & 0.7180 & 0.0830 & -0.0318 \\ 3.4109 & -2.5232 & 0.7193 & 0.0835 & -0.0319 \\ 3.4041 & -2.5152 & 0.7177 & 0.0834 & -0.0318 \end{bmatrix}$$

# Example 8 cont.

The norm of $P$ is less than 1, therefore there is unique bounded solution of the system, moreover the solution is given by the equation

$$\mathbf{X}(n) = \sum_{s=0}^{\infty} \mathbf{B}_{n-s+1}^{n} \Xi_{n-s}$$

for the fitted parameters $b_j(n)$.