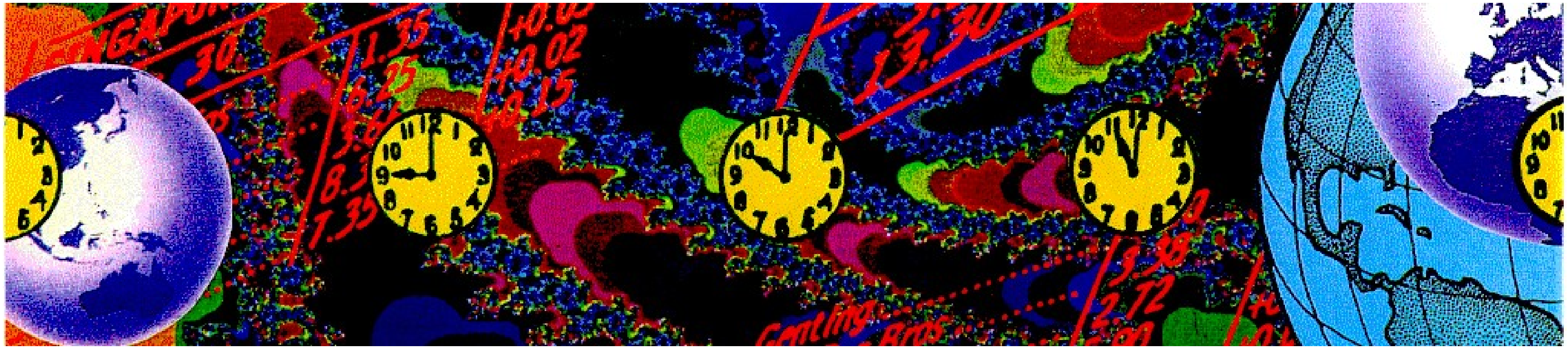
Stochastic volatility model of Heston and the smile

Rafał Weron

Hugo Steinhaus Center
Wrocław University of Technology
Poland

In collaboration with:
Piotr Uniejewski (LUKAS Bank)
Uwe Wystup (Commerzbank Securities)



Agenda

1. FX markets and the smile
2. Heston's model
3. Calibration

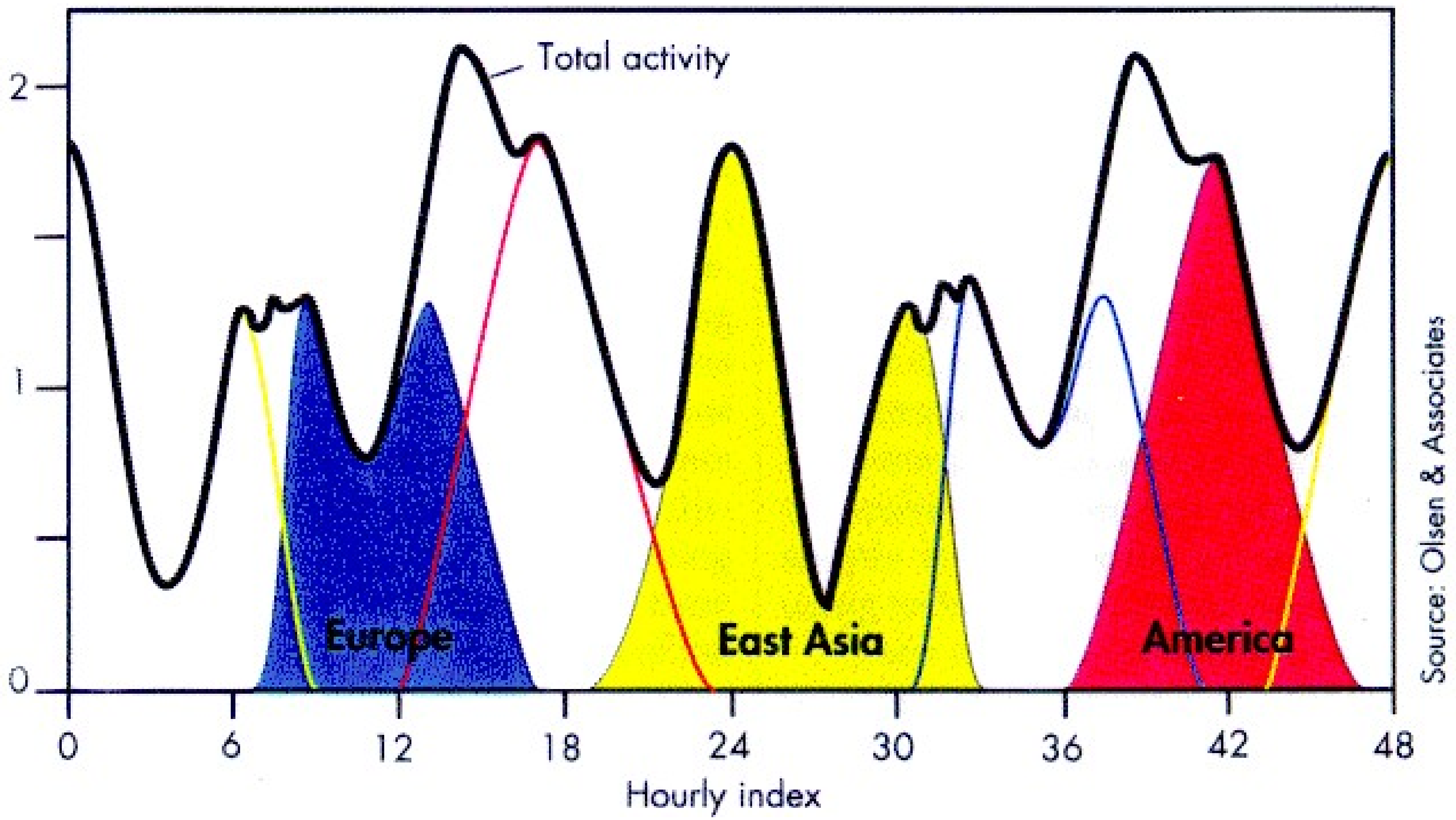
FX markets

EUR/USD and USD/JPY are two of the most liquid underlying markets with trading in:

- Spot/forward (ca. 90% of activity, very small margins)
- Vanilla options (9%, small margins)
- Exotic options (1%, potentially high margins)

Global markets

USD/JPY market activity



Black-Scholes type formula

- Assumes that asset prices follow GBM:

$$dS_t = S_t(\mu dt + \sigma dB_t) \tag{1}$$

- European FX call option price (Garman and Kohlhagen, 1983):

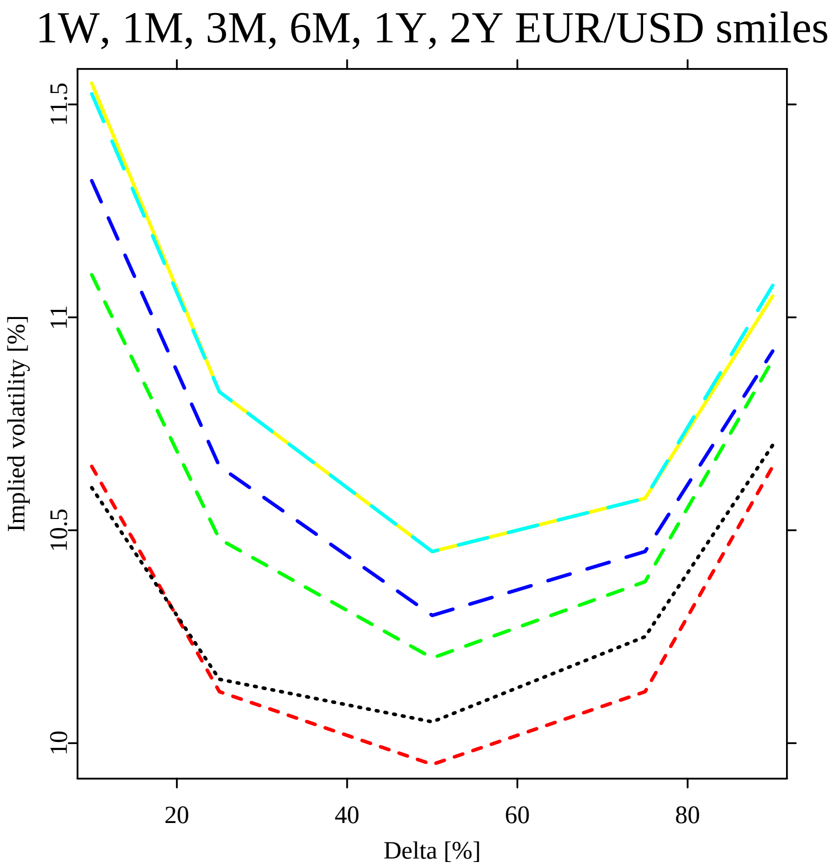
$$C_t = S_t e^{-r_f \tau}\Phi(d_1) - K e^{-r\tau}\Phi(d_2),$$

where $d_1 = \frac{\log(S_t/K)+(r-r_f+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$, $d_2 = d_1 - \sigma\sqrt{\tau}$

BS formula is flawed

- Implied volatility σ_i is the volatility that equates the BS price:
  $\mathrm{BS}(S_t, K, r, \sigma_i, \tau) = \text{Option market price}$
- Model implied volatilities for different strikes and maturities are not constant
- Volatility smile or smirk/grin is observed

The smile



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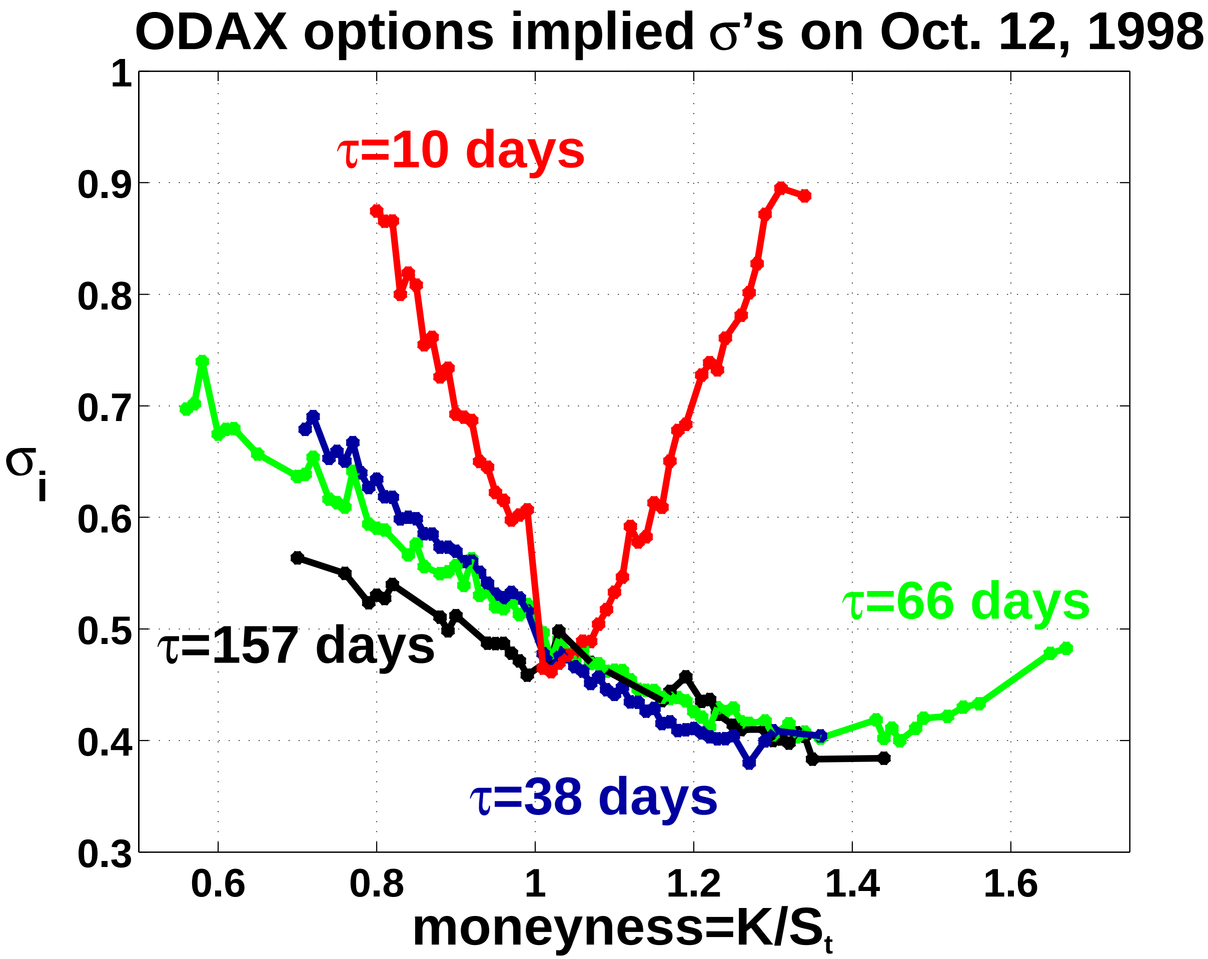
1W (black), 1M (red), 3M (green), 6M (blue), 1Y (cyan), and 2Y (yellow) EUR/USD implied volatility smiles on July 1, 2004

The smirk/grin



Correcting the BS formula (1/3)

- Allow the volatility to be a deterministic function of time (Merton, 1973): $\sigma = \sigma(t)$
- Explains the different σ_i levels for different τ's, but cannot explain the smile shape for different strikes

Correcting the BS formula (2/3)

- Allow not only time, but also state dependence of σ (Dupire, 1994; Derman and Kani, 1994; Rubinstein, 1994): $\sigma = \sigma(t, S_t)$
- Lets the local volatility surface to be fitted, but cannot explain the persistent smile shape which does not vanish as time passes

Correcting the BS formula (3/3)

- Allow the volatility coefficient in the BS diffusion equation (1) to be random: $\sigma = \sigma_t$
- Pioneering work of Hull and White (1987), Stein and Stein (1991), and Heston (1993) led to the development of stochastic volatility models

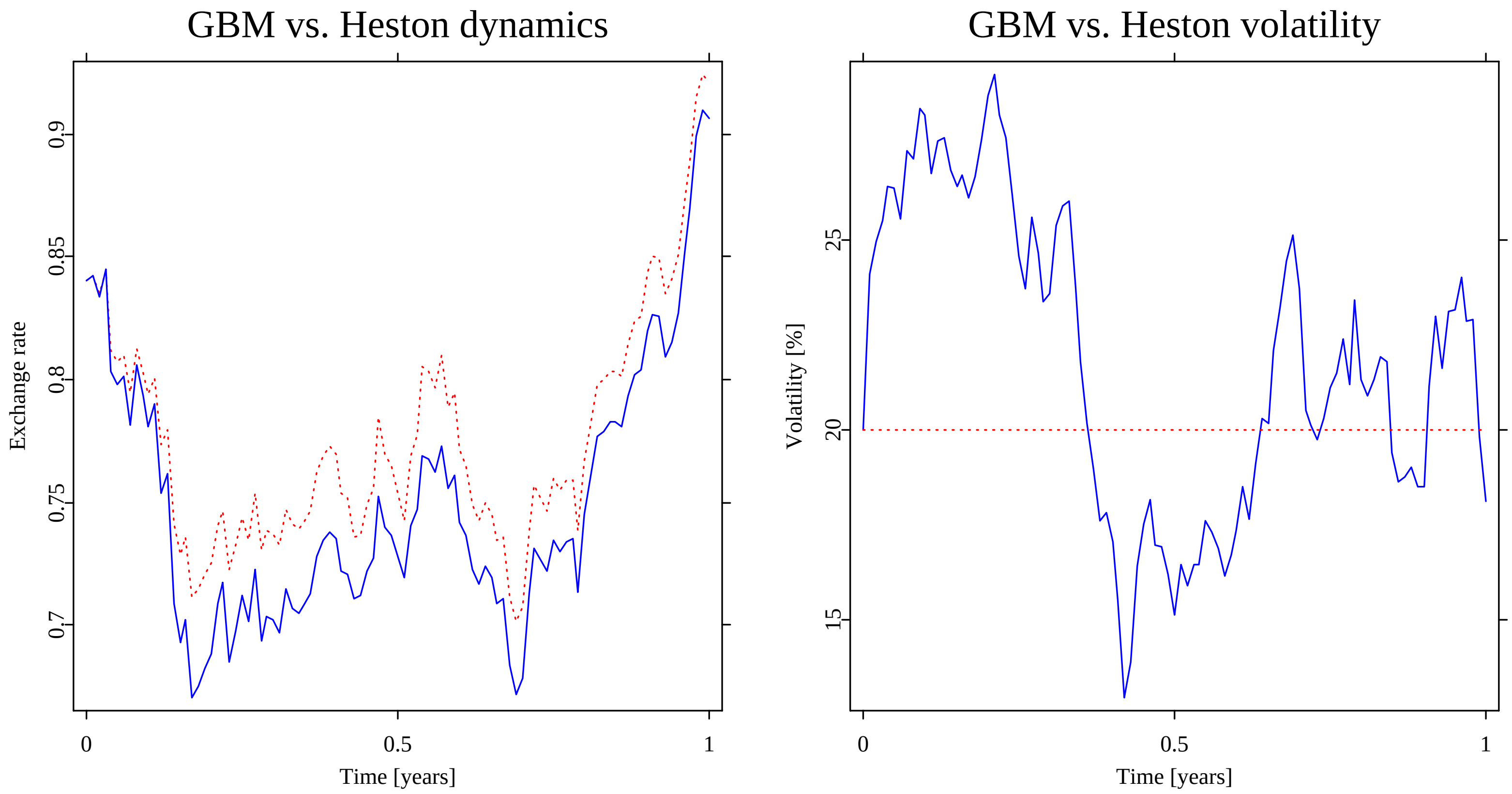
Heston's model

$$dS_t = S_t \left(\mu \, dt + \sqrt{v_t} dW_t^{(1)} \right), \tag{2}$$

$$dv_t = \kappa(\theta - v_t) \, dt + \sigma \sqrt{v_t} dW_t^{(2)}, \tag{3}$$

$$dW_t^{(1)} dW_t^{(2)} = \rho \, dt \tag{4}$$

- Variance process (3) is non-negative and mean-reverting (as observed in the markets)
- It has CIR dynamics



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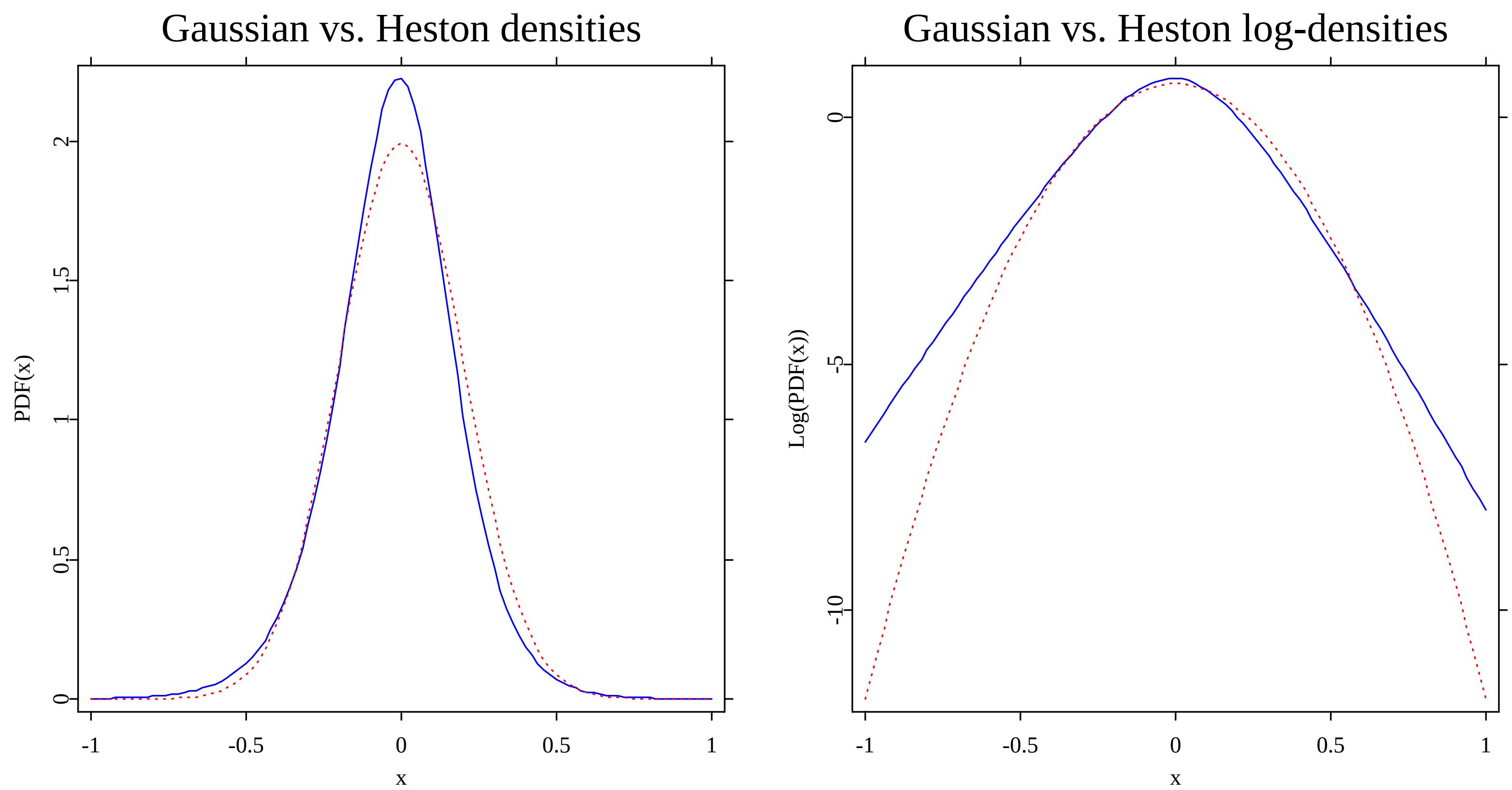
GBM vs. Heston: $\rho = -0.05$, initial (=GBM) variance $v_0 = 4\%$, long term var. $\theta = 4\%$, speed of mean reversion $\kappa = 2$, vol of vol $\sigma = 30\%$



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Unlike Gaussian tails, tails of Heston's marginals are exponential: log-densities resemble hyperbolas (Dragulescu and Yakovenko, 2002)

Option pricing in Heston's model

- PDE for the option price can be solved analytically using the method of characteristic functions (Heston, 1993)

- Closed-form solution for vanilla options:

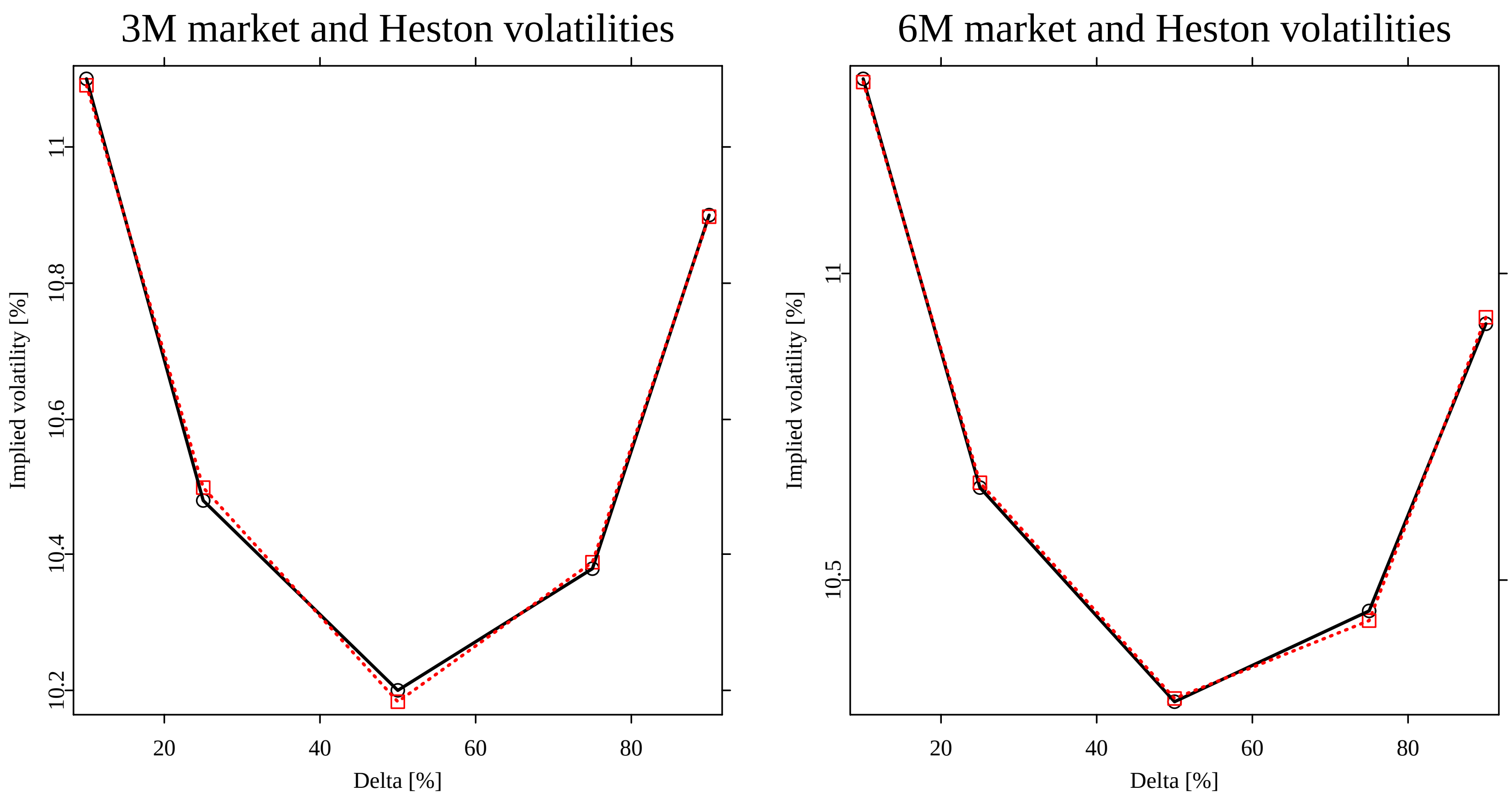
$$
\begin{aligned}
h(t) &= \text{HestonVanilla}(\kappa, \theta, \sigma, \rho, \lambda, r_d, r_f, v_0, S_0, K, \tau) \\
&= e^{-r_f \tau} S_t P_+(\phi) - K e^{-r_d \tau} P_-(\phi) \qquad (5)
\end{aligned}
$$

Calibration

1. Look at a time series of historical data:
   - Use GMM, SMM, EMM, or Bayesian MCMC to fit the price process
   - Fit empirical distributions of returns to the marginal distributions
   - Cannot estimate the market price of risk λ
2. Calibrate the model to derivative prices or better to the volatility smile

Calibration to the smile

- Take the smile of the current vanilla options market as a given starting point
- Find the optimal set of model parameters for a fixed τ and a given vector of market BS implied volatilities $\{\hat{\sigma}_i\}_{i=1}^{n}$ for a given set of delta pillars $\{\Delta_i\}_{i=1}^{n}$
- No need to worry about estimating λ as it is already embedded in the market smile



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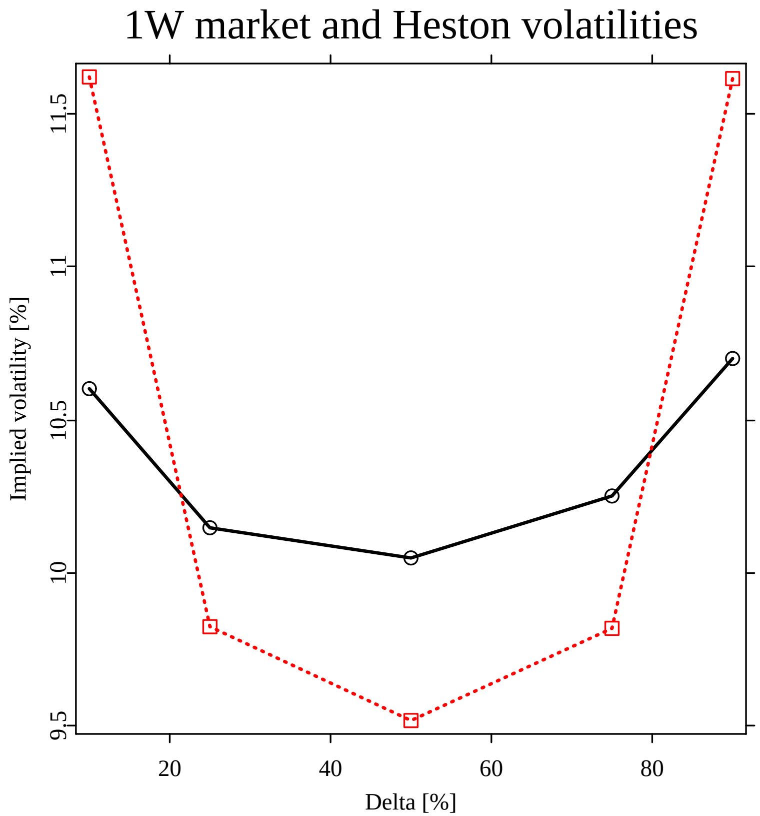
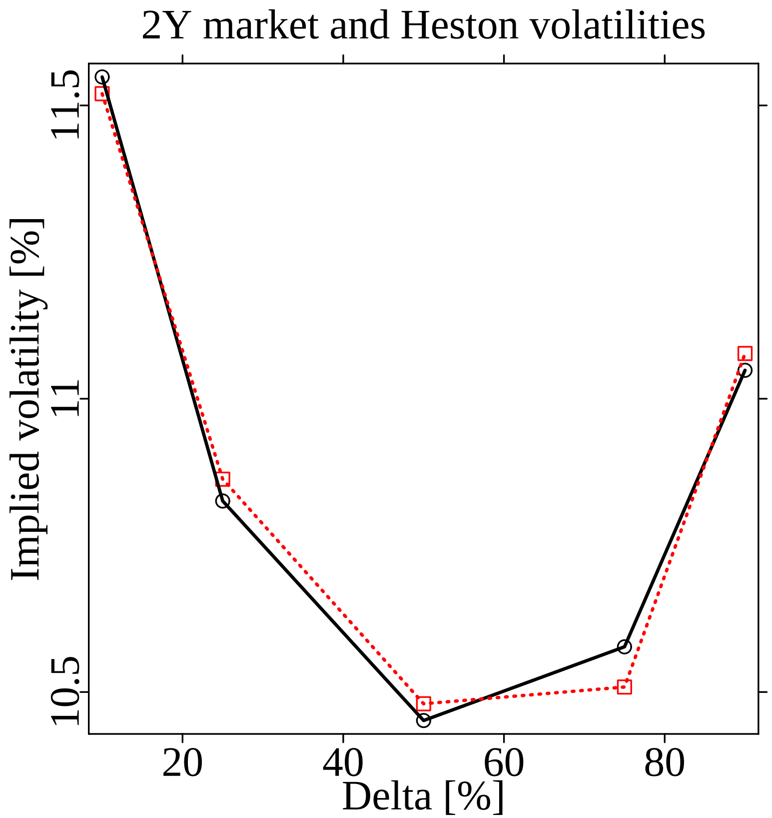
EUR/USD volatility surface on July 1, 2004: the fit is very good for maturities between three and eighteen months





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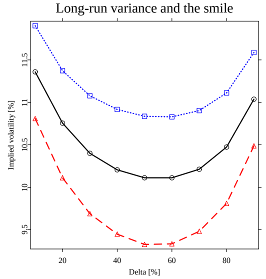
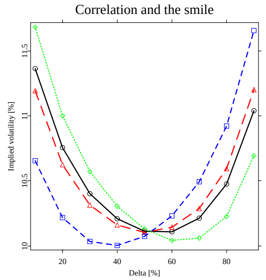
Unfortunately, Heston's model does not perform satisfactorily for short maturities and extremely long maturities

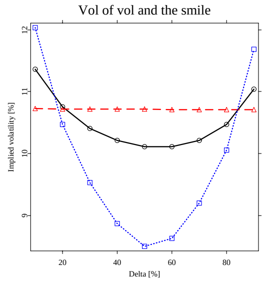
Only 3 parameters to fit



$\theta(\approx v_0)$ – ATM level of the smile



ρ – skew (quoted as risk reversals)



$\sigma(\approx \kappa)$ – convexity (butterflies)

Application

1. Calibrate the model to vanilla options
2. Employ it for pricing exotics, like one-touch or barrier options (finite difference, Monte Carlo)

References

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2. Weron, R. and Wystup, U. (2005) Heston's model and the smile, *in* P. Cizek, W. Härdle, R. Weron (eds.) *Statistical Tools for Finance and Insurance*, Springer.

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